AL–CAPODE: ALgorithms in Computer Assisted Proofs for Ordinary Differential Equations

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- **When?** spring 2026 (4 – 6 months)

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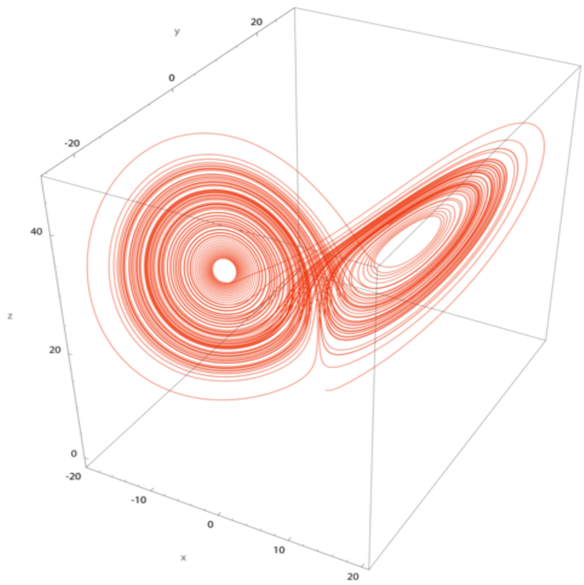

1 Scientific Context and Motivation

Differential equations are ubiquitous across scientific disciplines, from physics and engineering to biology and economics. While their definition is simple, they can encode highly complex dynamical behaviors, making their rigorous analysis both crucial and challenging. In recent years, computer-assisted proofs have revolutionized the field by solving long-standing open problems and conjectures that were previously out of reach for traditional pen-and-paper methods [9]. Notable examples are the proof of the existence of the Lorenz attractor by Tucker [8] (depicted above), or the very recent announced proof of non-uniqueness of Leray-Hopf solutions for the Navier-Stokes equations [4]. While the remarkable successes of recent years are the result of a fairly general approach, we are still a long way from *automatic processing* of these classes of dynamic systems, and the necessary numerical calculations are not performed using *formal tools* that can provide the level of safety that one would expect for these computer-assisted proofs. The aim of the project presented here is to make significant progress on the first issue.

2 State of the Art

A fairly general method [9, 7] [1, Chap. 1] for dealing with these problems has emerged over the past thirty years, particularly through a Newton-like a posteriori approach in suitable function spaces. This approach provides rigorous bounds on solutions and enables the verification of existence and uniqueness of solutions to differential equations. It consists of the following steps:

- **Formulate** the differential problem as a zero-finding problem for an operator $\mathscr{F}$. For example, the map $y = \exp$ is the unique zero y of

$$\mathscr{F} : y \mapsto \begin{pmatrix} y' - y \\ y(0) - 1 \end{pmatrix}; \tag{1}$$

- **Numerically solve** this problem with standard routines from numerical analysis, to obtain an approximation $\tilde{y}$ of the solution y under the form of a finite sum $\tilde{y}(x) = \sum_{k=0}^{n} a_k f_k(x)$ in a suitable basis for functions (f_k) (e.g., $f_k(x) = x^k$ for Taylor approximations, or $f_k(x) = e^{ikx}$ for periodic approximations);

- **A posteriori validate** this candidate approximation $\tilde{y}$ by computing a rigorous error bound on $\|\tilde{y} - y\| := \max_{x\in I} |\tilde{y}(x) - y(x)|$ over the interval of interest I. For this, the zero-finding problem $\mathscr{F}(y) = 0$ must be reformulated as a *fixed-point* problem $\mathscr{N}(y) = y$. Then Banach's fixed-point theorem yields the desired error bound, provided one can rigorously prove that $\mathscr{N}$ is contracting around $\tilde{y}$.

The construction of this fixed-point operator $\mathscr{N}$ notably requires to compute an explicit linear operator $\mathscr{A}$ approximating the inverse of $\mathscr{F}$ (or its linearization around $\tilde{y}$). To do so, a common approach in [9] [1, Chap. 1] consists in seeing $\mathscr{F}$ as an infinite matrix acting on sequences of coefficients (a_k) representing functions in the basis (f_k). $\mathscr{A}$ is then computed by inverting a finite-dimensional truncation of this matrix. However, the computational complexity of this approach remains a significant challenge, as investigated in [3]. This is due to the potentially large truncation index needed to compute a sufficiently accurate $\mathscr{A}$.

An alternative approach, proposed in [2], focuses on solving linear ordinary differential equations (ODEs) with prescribed initial conditions. Instead of computing $\mathscr{A}$ by finite-dimensional truncation and inversion, it relies on the following two ingredients:

- **an analytic formula** expressing the inverse operator. In the exp example above, $\mathscr{F}$ in (1) is inverted as

$$\mathscr{F}^{-1}(h,c) = y : x \mapsto \left(c + \int_0^x h(t)\exp(-t)\mathrm{d}t\right)\exp(x).$$

- **an effective approximation** $\mathscr{A}$ by replacing analytic functions appearing in this explicit formula by computable approximations, e.g., finite series in the basis (f_k). In the example above, $\exp(x)$ and $\exp(-x)$ are replaced by approximations $\varphi(x)$ and $\psi(x)$ to define

$$\mathscr{A}(h,c) = y : x \mapsto \left(c + \int_0^x h(t)\psi(t)\mathrm{d}t\right)\varphi(x),$$

with which explicit and rigorous computations are possible.

By exploiting the structure of the inverse rather than seeing it as an unstructured infinite matrix, this method features significantly better complexity and can solve harder instances.

3 Internship Project

The goal of this internship is to explore the generalization of this method to broader and more ambitious classes of differential equations, by addressing the following questions:

- Can the method of [2] be extended to **nonlinear ODEs**, by simply linearizing the operator $\mathscr{F}$ around the approximate solution $\tilde{y}$?

- How can this approach be generalized to **boundary value problems** or **periodic solutions of periodic ODEs**? Indeed, the existence of a solution is not always guaranteed.

- Is it possible to adapt this method to **delay differential equations**, which are of growing importance in modeling real-world phenomena? Such an equation, of the form

$$y'(x) = f(t, y(x), y(x-\tau)),$$

must come with an *infinite-dimensional* initial condition, namely the values of y over $(-\tau, 0]$.

- Depending on the differential equation, the type of initial conditions and the interval of definition (compact, infinite or semi-infinite), what is the **most suitable basis** (f_k)?

A proposed roadmap for this work is the following:

- Review the existing literature on computer-assisted proofs for differential equations and the various algorithms to compute rigorous solutions.
- Investigate the theoretical and practical challenges of extending the method of [2] to the above-mentioned classes of problems.
- Analyze the computational complexity of the resulting algorithms and implement them using, for example, the Arb/FLINT library for validated numerics [5][1].
- Test the algorithms on instances of problems from the literature: Can we compute the solutions faster? Can we extend the range of parameters over which the solution can be computed in reasonable time? A possible challenge would be to validate some of the works about chaos vs. long-term stability of the solar system by Laskar (see, e.g., [6], and [10] for a recent tentative of validation).

Depending on the progress and results, this work may naturally extend into a PhD project focused on the development of efficient algorithms and libraries dedicated to computer-assisted proofs for differential equations and dynamical systems.

4 Prerequisites

This internship targets Master students in computer science and/or mathematics. It requires typical undergraduate knowledge in mathematics – especially in analysis for differential equations – and computer science - computer algebra, computer arithmetic or approximation theory -, as well as an interest in code development.

References

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[1] `https://flintlib.org/`