Validated Numerical Software For Algebraic Curves With Singularities

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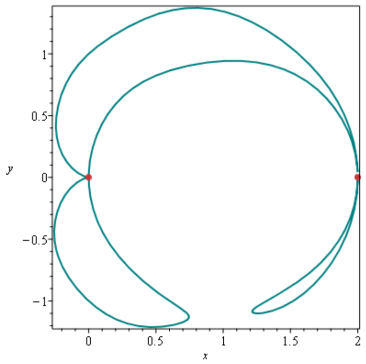

Validated numerics is the art of designing efficient numerical algorithms, yet *reliable* ones, i.e. with guaranteed error bounds encompassing all sources of errors: uncertain data, rounding errors, discretization, etc. The goal is to provide scientists in a broad sense with a "certified pocket calculator". This includes engineers working on safety-critical applications, but also a novel generation of mathematicians using computers to prove their theorems.

The goal of this internship is to design and implement validated algorithms to compute with *algebraic curves*, which arise in many branches of science. More specifically, we are interested by the difficult case of *singularities*, which may cause catastrophic numerical errors if not dealt properly with. These achievements will later allow us to treat currently unreachable applications in computer algebra, physics and robotics.

I. Scientific Context

Algebraic curves are a fundamental mathematical object with surprisingly many applications. Just within the context of this internship, this ranges from pure algebraic geometry (the crucial Abel-Jacobi map of a Riemann surface [4]) to nonlinear waves in physics (e.g., nonlinear Schrödinger, Korteweg-de Vries and Kadomtsev- Petviashvili equations [10, 9, 17]), computer-aided design and robotics (connectivity queries for motion planning [7, 11]). From a mathematical point of view, they are often defined by an *implicit* polynomial equation $P(x, y) = 0$. For the above-mentioned applications however, it is essential to be able to compute *explicitly* with them, notably parameterizations $x \mapsto y(x)$, intersections or contour integrals along them.

A wide range of *exact* algorithms have been designed by the computer algebra community for those problems [24, 1]. Unfortunately, larger instances, such as motion planning of a robot with many degrees of freedom, remain out of reach for such symbolic approaches. On the other side of the spectrum, efficient numerical methods relying on floating-point arithmetic have been developed to compute approximate solutions [20, 12]. Yet, numerical instability occurs when accumulation of errors results into inaccurate or meaningless solutions. This happens in particular when *singularities* in the curve (e.g., singular position of a robot [8]) are not detected and properly handled. This is a major pitfall since singularities are *not* an accident: they encode crucial information about the geometry of the curve and hence of the problem. Furthermore, in some specific contexts like safety-critical applications in engineering or computer-assisted proofs in mathematics (with computational parts of the proof carried out on a computer), relying on software without guarantees on numerical errors is not acceptable: a theorem is *not* true up to numerical errors, a medical robot is *not* safe up to erratic numerical behavior.

Validated numerics [22], built upon Moore's *interval analysis* [16], combines the advantages of both symbolic and numerical computation. It uses numerical set-valued representations of objects (intervals around numbers, tubes around functions, etc.) to exploit the efficiency of floating-point arithmetic while guaranteeing real mathematical statements: the solution, rather than being exactly represented, is contained in the computed set. Several works successfully applied those techniques to algebraic curves and computational algebraic geometry [13, 2, 14, 26]. Yet, a rigorous numerical treatment of curves around singularities remains a challenge that we propose to address in this internship using symbolic-numeric approaches described below. The resulting validated numerical software will allow us to treat in the future currently unreachable applications in computer algebra, physics and robotics detailed later.

II. A validated symbolic-numeric Newton-Puiseux algorithm

At a singular point where $\frac{\partial P}{\partial y}$ vanishes (let's say at $(x,y)=(0,0)$), there is no analytic parameterization $x \mapsto y(x)$ of the curve. The celebrated Newton-Puiseux algorithm [25, §IV.3] computes a formal solution $X \mapsto Y(X)$ in the form of a Puiseux series using *fractional* exponents to express the ramification of the curve at this singularity:

$$Y(X) = \sum_{n=n_0}^{+\infty} a_n X^{\frac{n}{e}} \quad \text{with} \quad n_0 \in \mathbb{Z} \quad \text{and} \quad e \in \mathbb{N}^*. \tag{1}$$

However, the performance of this symbolic algorithm is limited by the size of representations of the algebraic numbers a_n to be computed. The intern student will design a validated symbolic-numeric variant of Newton-Puiseux. A possible roadmap for this is to combine these two approaches:

- The symbolic-numeric approach advocated by one of us [18] consists in first executing Newton-Puiseux symbolically *but* modulo some small prime number, thus avoiding the representation size barrier. This allows us to infer the structure of the solution series (1), notably its ramification index e.
- To turn this strategy into an effective validated algorithms, the coefficients a_n must be numerically computed with rigorous error bounds in order to get a guaranteed parameterization of the curve all around the singularity. This will involve techniques from validated numerics, notably *fixed-point a posteriori validation* [23, 5], where rigorous error bounds are obtained afterwards from the application of a suitable fixed-point theorem. Elementary subproblems to tackle with such tools are the validation of multiple roots of univariate polynomials [27] and a generalized Hensel lifting procedure [19].

After designing his algorithm, the intern student will carefully analyze its complexity and compare it to the classical (i.e., symbolic) Newton-Puiseux. We expect a significant improvement, raising substantial hope to tackle currently unreachable applications.

Besides theoretical results, another important part of the work is a neat implementation of this validated symbolic-numeric Newton-Puiseux algorithm. For this, one can rely on dedicated validated numerics libraries (e.g., MPFI, Arb [15]). Depending on time and the specific skills of the student, further implementation-related directions can be explored: a formally verified implementation in the Coq theorem prover [3] using the ApproxModels library [6], or a high performance computing (HPC) implementation exploiting parallelism in the algorithms.

III. The Future: Challenges in Computational Algebraic Geometry

The algorithmic and implementation work realized during the internship will be the base of future challenging applications in real and complex algebraic geometry where analyzing singularities is a key step, and for which existing approaches show their limits.

- *Connectivity queries* are essential for motion planning in robotics [7, 11]. They can be handled by computing a *roadmap* of the algebraic variety. Efficient Puiseux series for the real branches of curves will improve this approach near singularities, which are currently an important bottleneck for practical applications.
- The *Abel-Jacobi map* [4, §1] is a construction of algebraic geometry linking essential information of a complex algebraic curve to computational data, namely contour integrals along paths connecting singularities. Being able to compute such objects efficiently and rigorously is a major step toward proofs of existence of particular solutions to nonlinear wave equations in physics [10, 9, 17]: KdV (Korteweg-de Vries), KP (Kadomtsev-Petviashvili) and NLS (nonlinear Schrödinger). This also has numerous applications in computer algebra, e.g. integrating algebraic functions [21].

IV. Internship Candidate and Advisors

The internship will be co-advised by François Boulier, Adrien Poteaux and Florent Bréhard. The three of them are members of the CFHP team (Computer Algebra and HPC) in the CRIStAL research unit at Université de Lille.

The candidate must have preferably a mixed background in computer science (scientific/numerical programming, algorithmics) and mathematics (algebra, complex analysis), and a taste for *computational* mathematics towards applications. Previous experiences in computer algebra, HPC and/or formal proof will also be considered.

V. References

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